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The nonlinear evolution of the inviscid secondary instability of streamwise vortex structures

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The weakly nonlinear evolution of an inviscid marginally unstable wave growing on a boundary layer supporting a streamwise vortex structure is investigated. The nonlinear growth of the wave is found to be controlled by the diffusion layer located at the edge of the critical layer associated with the wave. The evolution equation is found to depend on the upstream history of the wave and the solution of the equation suggests that the wave either restructures the mean state so as to make it stable or develops a singularity at a finite distance downstream of the point of neutral stability.

1. Introduction

Streamwise vortices are known to be an important ingredient of the transition process in boundary-layer flows over both curved and flat walls. In the case of flat walls, the streamwise vortices are an initially passive product of wave interactions (see, for example, Hall & Smith 1991), while for curved walls the vortex can be the primary instability of the undisturbed flow. In recent years there has been much interest in the breakdown process of such vortices. In the context of Görtler vortices, the experiments of Swearingen & Blackwelder (1987) show that the initially steady vortex structure caused by wall curvature undergoes a secondary bifurcation to an unsteady three-dimensional flow. This secondary instability can be of either the sinuous or varicose type. In flat plate boundary layers the secondary instability is invariably found to be of the varicose type.

The first attempt to understand the breakdown process for Görtler vortices was given by Hall & Seddougui (1989). In that calculation, the small-wavelength Görtler vortex flow, investigated by Hall & Lakin (1988), was localized in the shear layers, trapping the region of vortex activity. Analytical progress for this case is made possible by the assumption of small-vortex wavelength and Hall & Seddougui (1989) showed that wavy (i.e. sinuous) modes of instability are possible and are localized near the top and bottom of the vortices. No mode of the varicose type was found to be unstable. If such a mode exists at small-vortex wavelengths then it is presumably not trapped in the region where the vortex activity decays to zero.

In order to understand the origin of this secondary instability mechanism, Hall & Horseman (1991) investigated the inviscid linear instability of a Görtler vortex. The instability analysis for such a flow is greatly simplified by the fact that the streamwise

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velocity component of the flow in the presence of a Görtler vortex remains an order of magnitude larger than the normal and spanwise velocity components. This means that an inviscid disturbance to the vortex flow satisfies a two-dimensional form of the Rayleigh equation, dependent only on the streamwise velocity component of the vortex. Note here that the length and time scales of the inviscid mode are relatively short so that non-parallel effects are formally negligible in the leading-order inviscid instability analysis. The modified two-dimensional Rayleigh equation found by Hall & Horseman (1991) was simultaneously found in the context of vortex wave interaction theory by Hall & Smith (1991). The numerical solution of the eigenvalue problem associated with the modified Rayleigh equation was discussed by Hall & Horseman (1991), who found that both sinuous and varicose modes become unstable as the vortex develops in the streamwise direction. The question of which mode is the most unstable is a function of the history of the vortex, its wavelength and the Görtler number. However, Hall & Horseman (1991) were able to obtain quantitative agreement with the experimental measurements of Swearingen & Blackwelder (1987). Subsequently, the instability problem was examined again by Li & Malik (1995) who found that Hall & Horseman (1991) had missed the most unstable varicose mode. Note that this latter mode is in fact the second most unstable overall and that otherwise the results of Li & Malik (1995) are consistent with those of Hall & Horseman (1991).

Related investigations of the problem discussed above have been given by Liu & Domardzki (1993) and Yu & Liu (1994). In the former paper, a direct Navier–Stokes simulation of transition in the Görtler problem was carried out for parallel flows, and the results obtained are in broad agreement with those of Hall & Horseman (1991). Yu & Liu (1994) reconsidered the instability problem of Hall & Horseman (1991) but retained some viscous terms in their approximation. Yu & Liu (1994) are critical of Hall & Horseman (1991) because of their neglect of viscosity. However, this criticism is perhaps surprising since viscous effects are certainly a second-order effect and in any case the equations solved by Yu & Liu (1994) retain only *some* of the second-order effects. In effect, the criticism of Yu & Liu (1994) is equivalent to arguing that the Orr–Sommerfeld equation, rather than the Rayleigh equation, should be used to describe inviscid instabilities of parallel flows.

In this paper we shall describe the evolution of the inviscid mode found by Hall & Horseman (1991). This will be done using viscous-critical-layer and diffusion-layer theories in the context of a weakly nonlinear instability theory. In particular, we shall consider the evolution of a mode near the critical streamwise location, where the vortex structure has developed sufficiently to (first) render the now three-dimensional boundary-layer flow unstable to inviscid modes (note that, for example, incompressible two-dimensional Blasius boundary-layer flow does not support inviscid instability). At such a location, the flow is marginally unstable and we can consider the evolution of the most dangerous (important) mode. Related work has been carried out by Wu (1993) and Smith *et al.* (1993).

Although the analysis we give is for vortex flows generated by wall curvature, it is valid for any flow where one of the velocity components depends on two spatial variables and is larger than the other two components. For such more general flows the periodicity in the spanwise direction which we assume in this paper must be replaced by an appropriate condition in order to derive the required solvability condition. The procedure adopted in the rest of this paper is as follows. In § 2 we formulate the problem to be considered. In § 3 we determine the outer solution of the perturbation equations and the required form of the solvability condition, while the critical layer is discussed in § 4. At the edge of the critical layer, a diffusion layer is required in order to account for the mean-flow correction; this layer is discussed in § 5 and the evolution equation is derived; the solution of the evolution equation and some conclusions are given in § 6.

2. Formulation

We consider the flow of a viscous fluid of kinematic viscosity ν over a wall of variable curvature. If U_0 is a typical value of the fluid speed at infinity and l is a typical length scale in the flow direction, then, by defining a Reynolds number $Re = U_0 l/\nu$, the non-dimensionalised Navier–Stokes equations for an incompressible flow may be written in the form

$$\left.\begin{aligned} u_x + v_y + w_z &= 0,\\ u_t + uu_x + vu_y + wu_z &= -p_x + Re^{-1}(u_{xx} + u_{yy} + u_{zz}),\\ v_t + uv_x + vv_y + wv_z &= -p_y + Re^{-1}(v_{xx} + v_{yy} + v_{zz}),\\ w_t + uw_x + vw_y + ww_z &= -p_z + Re^{-1}(w_{xx} + w_{yy} + w_{zz}). \end{aligned}\right\} \tag{2.1}$$

We define new variables Y, Z by writing

$$y = Re^{-1/2}\, Y, \quad z = Re^{-1/2}\, Z, \tag{2.2}$$

and assume a large-Reynolds-number ($Re \gg 1$) three-dimensional boundary-layer flow (i.e. Blasius flow plus streamwise vortex) of the form

$$(u, v, w, p) = (\bar{u}(x, Y, Z), Re^{-1/2}\bar{v}(x, Y, Z), Re^{-1/2}\bar{w}(x, Y, Z), p_\infty + Re^{-1}\,\bar{p}(x, Y, Z)).$$

This flow can be generated by several mechanisms but the most obvious one is wall curvature. In that case, x, y in (2.1) measure distance along and normal to the wall, and curvature terms must be inserted into those equations. We then find that the flow is determined by

$$\left.\begin{aligned} \bar{u}_x + \bar{v}_Y + \bar{w}_Z &= 0,\\ \bar{u}_t + \bar{u}\bar{u}_x + \bar{v}\bar{u}_Y + \bar{w}\bar{u}_Z &= \bar{u}_{YY} + \bar{u}_{ZZ},\\ \bar{v}_t + \bar{u}\bar{v}_x + \bar{v}\bar{v}_Y + \bar{w}\bar{v}_Z + \tfrac{1}{2}G\bar{u}^2 &= -\bar{p}_Y + \bar{v}_{YY} + \bar{v}_{ZZ},\\ \bar{w}_t + \bar{u}\bar{w}_x + \bar{v}\bar{w}_Y + \bar{w}\bar{w}_Z &= -\bar{p}_Z + \bar{w}_{YY} + \bar{w}_{ZZ}, \end{aligned}\right\} \tag{2.3}$$

which must be solved in conjunction with suitable boundary conditions such as 'no-slip' at the wall (solid boundary) and with the solution tending towards a uniform flow at infinity (far from the solid boundary). The parameter G appearing in (2.3) is the Görtler number. The extra curvature terms to be inserted in (2.1) play no role in the nonlinear inviscid instability problem to be investigated here, so for convenience we choose not to insert them. Note also that, from Hall & Bennett (1986), the Görtler equations are perhaps more conveniently derived by starting with (2.1) and then making an appropriate Prandtl transformation. Again, if this route is followed, the extra terms to be inserted into (2.1) play no role in the following analysis. Thus, in the present investigation, it is sufficient for us to use (2.1) to describe the nonlinear state perturbed around the incoming vortex flow given above. Whether the vortex field is generated by curvature, turbulence screens upstream or a localized bump at the wall is irrelevant in the following discussion.

In general, the solution of (2.3) is a numerical task (see Hall 1988) and it turns out that nonlinear effects stabilize the growth of Görtler vortices; however, when a large *spanwise* wavenumber assumption is made, Hall & Lakin (1988) demonstrated that much analytical progress can be made towards the solution of these equations.

Let us first recap the linear inviscid stability problem for this three-dimensional boundary-layer flow. In the neighbourhood of a point x_0, the flow is perturbed by a small inviscid disturbance proportional to

$$E = \exp[\mathrm{i}(\alpha X - \Omega T)], \tag{2.4a}$$

where

$$x = x_0 + Re^{-1/2} X, \quad t = Re^{-1/2} T, \tag{2.4b}$$

and α, Ω are the (streamwise) wavenumber and frequency of the linear inviscid secondary instability. The expansions for the velocities and pressure are

$$(u, v, w, p) = (\bar{u}, Re^{-1/2}\,\bar{v}, Re^{-1/2}\,\bar{w}, p_\infty + Re^{-1}\,\bar{p}(x, Y, Z)) + \epsilon((\hat{u}, \hat{v}, \hat{w}, \hat{p})E + \text{c.c.}), \tag{2.5}$$

where $\epsilon \ll 1$, 'c.c.' represents complex conjugate, barred quantities correspond to the three-dimensional boundary-layer flow and the disturbance quantities $\hat{u}, \hat{v}, \hat{w}, \hat{p}$ are, in particular, functions of x, Y and Z (but not X or T).

After a little algebra we find that the pressure perturbation $\hat{p}$ satisfies the modified Rayleigh pressure equation

$$\frac{\partial}{\partial Y}\left[\frac{\hat{p}_Y}{(\bar{u}-c)^2}\right] + \frac{\partial}{\partial Z}\left[\frac{\hat{p}_Z}{(\bar{u}-c)^2}\right] - \frac{\alpha^2 \hat{p}}{(\bar{u}-c)^2} = 0, \tag{2.6a}$$

with boundary conditions

$$\hat{p}_Y(y = 0) = 0, \quad \hat{p}(Y \to \infty) = 0, \tag{2.6b}$$

where $c = \Omega/\alpha$. In this paper we shall restrict our attention to solutions of this equation with $\hat{p}$ periodic in Z with the same Z-period as the underlying flow $\bar{u}$; in fact, without loss of generality, we choose this Z-period to be 2π. This equation was derived by Hall & Smith (1991), who were concerned with vortex-wave interactions, and by Hall & Horseman (1991) in the context of secondary instabilities of Görtler vortices. The eigenvalue problem for $c \equiv c(\bar{u}, x, \alpha)$ (temporal stability problem), associated with the partial differential system for $\hat{p}$, was first solved by Hall & Horseman (1991). Here we shall consider the more appropriate spatial instability problem in the presence of nonlinear effects.

Note that c is *not* a function of Z; if we consider neutral disturbances (those having c entirely real), equation (2.6*a*) is singular at $Y = Y_\mathrm{C} \equiv f(x, Z)$, (say) where $\bar{u} = c$. Thus, for three-dimensional boundary layers, the critical layer is 'wavy' in the sense that the location of the critical level (where the equation is singular) is a function of spanwise location Z. Note further that the neutral value of c, and hence $Y_\mathrm{C} \equiv f(x, Z)$, are *not* known in advance of a numerical solution of the eigenvalue problem (2.6).

In figures 1 and 2 we present some results from Hall & Horseman (1991) for their numerical solution of (2.6) in the context of secondary instabilities of Görtler vortices; the reader is referred to their paper for more details of the underlying three-dimensional flow used in the calculations. In figure 1 we show some results for the most dangerous odd mode of instability for a Görtler vortex flow. The shape of the curve is similar to that which would be obtained for an inflectional unidirectional

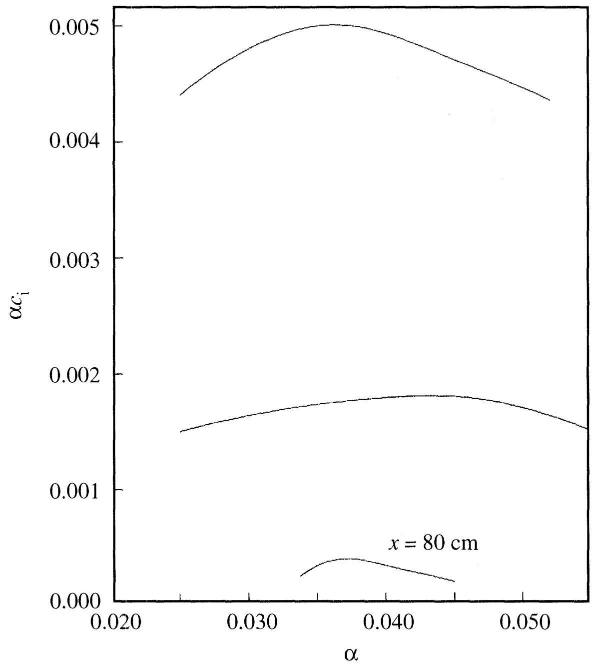

Figure 1. The growth rates of the two most dangerous odd modes at a position $x = 100$ cm from the leading edge; the lowest curve corresponds to the only unstable mode found at $x = 80$ cm (reproduced from Hall & Horseman (1991)).

velocity field by solving the standard Rayleigh equation. In figure 2 we show the normal and streamwise perturbation velocity components for an unstable mode. Note that the mode is concentrated towards the edge of the boundary layer. Since the mode is not neutral, it does not exhibit a critical-layer behaviour.

In fact, it is a difficult numerical procedure to solve (2.6*a*) in the neutral case (as the equation is singular at the critical level) and results are not yet available. However, it is clear from the results of Hall & Horseman (1991) that neutral modes of the eigenvalue problem (2.6) do exist (see, for example, their figure 3(*a*)—reproduced here as figure 1). That such neutral modes will be regular cannot be deduced so easily from their results; in the next section we argue that such neutral modes must be regular. The analysis presented in this paper is for the case of marginally unstable flows, i.e. we are assuming that the flow is stable until a certain downstream location, where the three-dimensional underlying flow has developed such that it first supports unstable linear inviscid modes governed by (2.6). This criterion is obviously met for the developing incompressible three-dimensional boundary-layer flow considered by Hall & Horseman (1991). Their results show that the ever-increasingly Görtler-vortex-dominated flow becomes more unstable in the downstream direction; however, upstream the Görtler vortices have not developed sufficiently and the flow is two dimensional (Blasius) at leading order and hence completely stable to such inviscid perturbations. Thus, by the so-called 'sandwich principle', there exists a streamwise location at which the flow first supports unstable linear inviscid modes governed by (2.6). Now let us consider the nonlinear problem and, as a starting point, we discuss the outer region away from the critical layer.

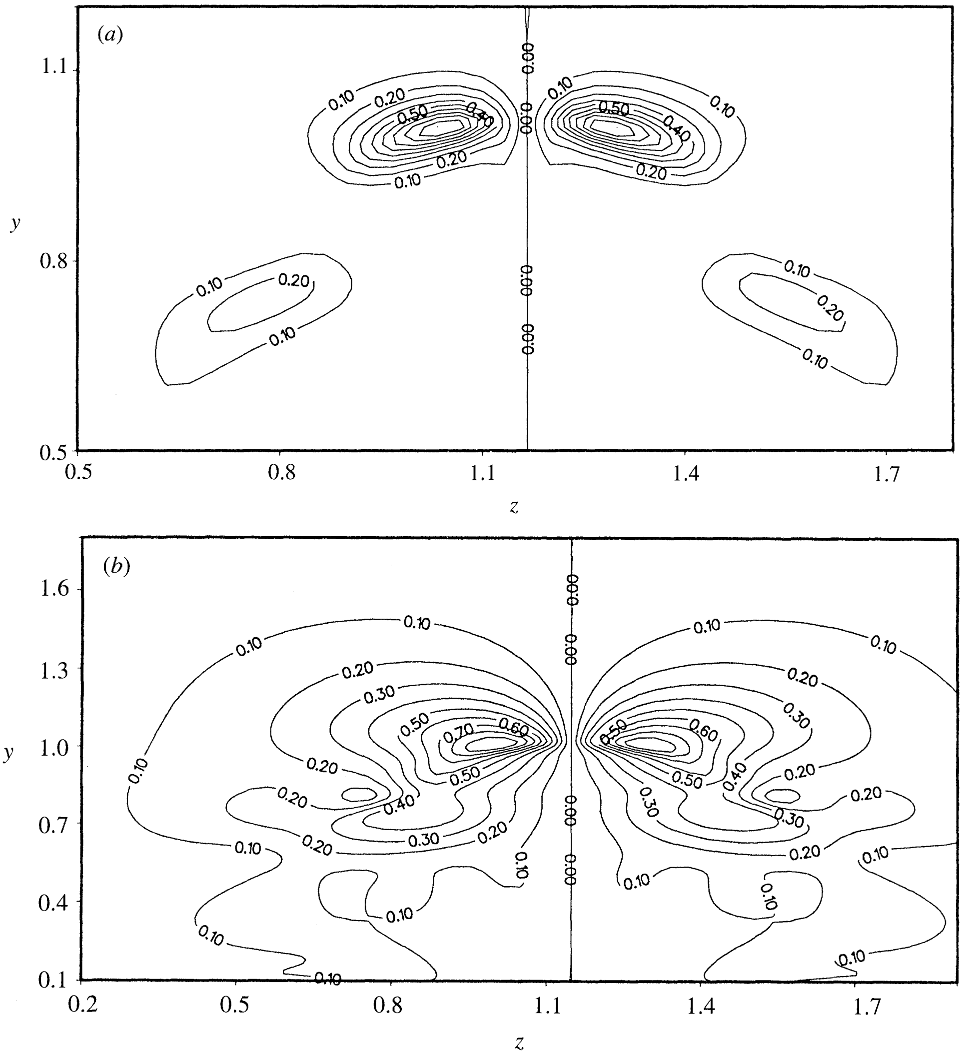

Figure 2. The contours of the streamwise and normal velocity components for the most dangerous odd mode at a position $x = 100$ cm from the leading edge: (*a*) contours of constant $|u|$; (*b*) contours of constant $|v|$ (reproduced from Hall & Horseman (1991)).

3. The outer solution for weakly nonlinear inviscid modes

In order to derive the desired nonlinear evolution equation for the amplitude of an inviscid disturbance mode, it is necessary to split the three-dimensional boundary-layer flow into distinct regions (layers), each corresponding to different dominant physical effects locally governing the inviscid disturbance (see figure 3).

Let us first consider the flow solutions in regions Ia,b, away from the critical layer, diffusion layer and solid boundaries (the wall), i.e. the bulk of the boundary layer. Our aim is to derive the solvability condition associated with an inhomogeneous form of (2.6). This solvability condition, together with expressions for certain 'jumps' in flow velocities in crossing the critical layer, (to be derived in the later subsections) will yield an evolution equation for the spatial evolution of the disturbance amplitude.

Before proceeding any further, let us consider the various streamwise length scales

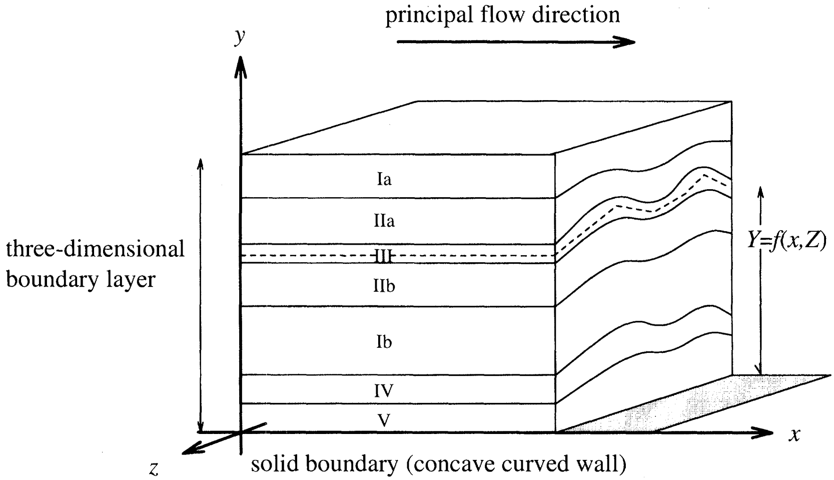

Figure 3. The various regions of the three-dimensional boundary-layer flow: Ia,b, outer flow; IIa,b, diffusion layers of relative thickness $O(Re^{-1/4}\,\mu^{-1/2})$; III, critical layer of relative thickness $O(Re^{-1/6})$; IV, outer wall layer of relative thickness $O(Re^{-1/6})$; V, inner wall layer of relative thickness $O(Re^{-1/4})$. Note that: (i) these thicknesses are given relative to the boundary-layer thickness $O(Re^{-1/2})$; and (ii) the critical and diffusion layers are distinct from the wall being centered around the level $Y = f(x, Z)$.

which are present in the analysis. In fact, we use multiple scales in the streamwise direction x:

$$x = x_0 + Re^{-1/2}\,\mu^{-1}\tilde{X} + \mu\tilde{x} + Re^{-1/2}\,X, \quad \mu \ll 1. \tag{3.1a}$$

Here x_0 now denotes the neutral x-station (in fact, x_0 is a point of marginal instability, as the flow first becomes unstable to linear inviscid disturbances here); $\mu\tilde{x}$ is a small change in x-location from x_0 (assuming $\mu\tilde{x} > 0$, then the flow is now slightly unstable: hence the weakly nonlinear analysis to follow); $Re^{-1/2}\,\mu^{-1}\tilde{X}$ is the wave-amplitude modulation scale; and $Re^{-1/2}\,X$ is the scale on which the waves oscillate. Note that we are considering the purely spatial stability problem and thus there are no slow time effects. In order to elucidate the effect of the non-parallelism of the underlying flow on the nonlinear evolution of the disturbance, we define the parameter m:

$$\mu = m\,Re^{-1/4}, \tag{3.1b}$$

so that (3.1a) can now be rewritten as

$$x = x_0 + \mu(\tilde{x} + m^{-2}\tilde{X}) + Re^{-1/2}\,X. \tag{3.1c}$$

We see that when μ reduces to $O(Re^{-1/4})$, corresponding to $m \sim O(1)$, the $\tilde{x}$-scale emerges at the same order as the wave-amplitude modulation scale and, hence, the non-parallelism of the underlying flow will affect the evolution of the disturbance (see Hall & Smith 1984).

We consider 'fixed-frequency' disturbances so that Ω is fixed; thus, the non-neutrality is *entirely* due to the change in x-location $\mu\tilde{x}$. Note that the developing three-dimensional boundary-layer, and hence the neutral streamwise-wavenumber α, are functions of x. In addition, it should be noted that, if we were considering a three-dimensional shear layer rather than a three-dimensional boundary layer, we would not consider Ω fixed. The non-neutrality would then be *entirely* due to the Ω perturbation, i.e. a Strouhal number perturbation.

Since the normal and spanwise velocity components of the unperturbed flow are relatively small, only the streamwise component $\bar{u}$ of the three-dimensional boundary-layer flow enters the problem to orders of concern; we write

$$\bar{u} = \bar{u}_0 + \mu(\tilde{x} + m^{-2}\tilde{X})\bar{u}_1 + \cdots, \tag{3.2a}$$

where $\quad \bar{u}_0(Y,Z) = \bar{u}(x_0,Y,Z), \quad \bar{u}_1(Y,Z) = \bar{u}_x(x_0,Y,Z).$

Further, we only need to consider two terms of the fundamental (proportional to $E^{\pm}$) and write

$$(\hat{u},\hat{v},\hat{w},\hat{p}) = (\hat{u}_1,\hat{v}_1,\hat{w}_1,\hat{p}_1) + \cdots + \mu(\hat{u}_2,\hat{v}_2,\hat{w}_2,\hat{p}_2) + \cdots. \tag{3.2b}$$

(a) The leading fundamental term

The leading-order problem for $(\hat{u}_1,\hat{v}_1,\hat{w}_1,\hat{p}_1)$ is a partial differential system in Y, Z and its solution may be written in the form

$$(\hat{u}_1,\hat{v}_1,\hat{w}_1,\hat{p}_1) = A(\tilde{X})(\hat{u}_1^{(1)},\hat{v}_1^{(1)},\hat{w}_1^{(1)},\hat{p}_1^{(1)})E + \text{c.c.}, \tag{3.3a}$$

where A is an amplitude function and $\hat{u}_1^{(1)}, \hat{v}_1^{(1)}, \hat{w}_1^{(1)}$ and $\hat{p}_1^{(1)}$ satisfy

$$\left.\begin{aligned} \mathrm{i}\alpha\hat{u}_1^{(1)} + \hat{v}_{1Y}^{(1)} + \hat{w}_{1Z}^{(1)} = 0, \quad \mathrm{i}\alpha(\bar{u}_0 - c)\hat{u}_1^{(1)} + \bar{u}_{0Y}\hat{v}_1^{(1)} + \bar{u}_{0Z}\hat{w}_1^{(1)} = -\mathrm{i}\alpha\hat{p}_1^{(1)},\\ \mathrm{i}\alpha(\bar{u}_0 - c)\hat{v}_1^{(1)} = -\hat{p}_{1Y}^{(1)}, \quad \mathrm{i}\alpha(\bar{u}_0 - c)\hat{w}_1^{(1)} = -\hat{p}_{1Z}^{(1)}. \end{aligned}\right\} \tag{3.3b}$$

These lead to the eigenvalue problem (2.6) (with $\hat{p}_1^{(1)}$ and $\bar{u}_0$ replacing $\hat{p}$ and $\bar{u}$, respectively) to determine the neutral solution at $x = x_0$. We fix $\hat{p}_1^{(1)}$ by the normalization

$$\hat{p}_1^{(1)} = \hat{p}_\infty(Z)\mathrm{e}^{-\alpha Y} \quad \text{as } Y \to \infty, \tag{3.3c}$$

where $\hat{p}_\infty(Z)$ is prescribed at some value of Z.

Following Hall & Smith (1991), let us now consider the behaviours of $\hat{u}_1^{(1)}$, $\hat{v}_1^{(1)}$, $\hat{w}_1^{(1)}$ and $\hat{p}_1^{(1)}$ near the critical layer at $Y = Y_c \equiv f(Z)$. Defining $s = Y - Y_c$ (so that $\partial_Z \to \partial_Z - f_Z\partial_s$), we write

$$\left.\begin{aligned} \bar{u}_0 &= c + \lambda(x_0,Z)s + \lambda_2(x_0,Z)s^2/2 + \lambda_3(x_0,Z)s^3/6 + \cdots,\\ \bar{u}_1 &= \bar{u}_{10}(x_0,Z) + \bar{u}_{11}(x_0,Z)s + \bar{u}_{12}(x_0,Z)s^2/2 + \cdots. \end{aligned}\right\} \tag{3.4}$$

The method of Frobenius gives $\hat{p}_1^{(1)}$ in the form

$$\hat{p}_1^{(1)} = \phi_1 + b_{1\pm}(Z)\phi_2, \tag{3.5a}$$

where

$$\phi_1 = \phi_{10}(Z) + \phi_{11}(Z)\cdot s + \phi_{12}(Z)\cdot s^2 + \phi_{13L}(Z)\cdot s^3\ln s + 0\cdot s^3 + \cdots, \quad \phi_2 = s^3 + \cdots. \tag{3.5b}$$

Note that $\phi_{10}(Z)$ and $b_{1\pm}$ can be determined from the numerical solution of the eigenvalue problem (2.6) (the $\pm$ corresponding to above and below the critical level, respectively); while, in particular,

$$\Delta\phi_{11} = f_Z\phi_{10Z}, \tag{3.6a}$$

$$2\lambda\Delta^2\phi_{12} = \lambda\Delta\phi_{10ZZ} - (2\lambda_Z + \lambda f_Z f_{ZZ})\phi_{10Z} - \alpha^2\lambda\Delta\phi_{10} \tag{3.6b}$$

$$\begin{aligned}6\lambda\Delta\phi_{13L} = {} & 4\lambda f_Z\phi_{12Z} + 2(3\lambda_2\Delta - 4f_Z\lambda_Z + 2\lambda f_{ZZ})\phi_{12} - 2\lambda\phi_{11ZZ} \\ & +2(2\lambda_Z - \lambda_2 f_Z)\phi_{11Z} + (2\alpha^2\lambda - 2f_Z\lambda_{2Z} + \lambda_2 f_{ZZ})\phi_{11} \\ & -\lambda_2\phi_{10ZZ} + 2\lambda_{2Z}\phi_{10Z} + \alpha^2\lambda_2\phi_{10},\end{aligned} \tag{3.6c}$$

where we have defined

$$\Delta = 1 + f_Z^2. \tag{3.7}$$

Here, following Hall & Smith (1991), we assume $\phi_{13L} = 0$, i.e. we assume that the right-hand side of (3.6*c*) is identically zero. If it were non-zero, at $O(Re^{-1/6})$ a jump in $\hat{v}$ across the sole critical layer would be induced and, as we are not looking at upper-branch scalings, there would be no wall layer effects to counteract this jump; hence, this jump and the coefficient of the logarithm term must be zero. Equations (3.6*a*) and (3.6*b*) determine the coefficients ϕ_{11} and ϕ_{12} in terms of $\phi_{10}(Z)$.

The velocities $\hat{u}_1^{(1)}$, $\hat{v}_1^{(1)}$ and $\hat{w}_1^{(1)}$ have the following properties near the critical layer:

$$\hat{u}_1^{(1)} = d_{1Z}s^{-1} + \cdots, \quad \hat{v}_{P1}^{(1)} \equiv \hat{v}_1^{(1)} - f_Z\hat{w}_1^{(1)} = d_2 + \cdots, \quad \hat{w}_1^{(1)} = -\mathrm{i}\alpha d_1 s^{-1} + \cdots, \tag{3.8a}$$

where

$$d_1 = -\phi_{10Z}/\alpha^2\lambda\Delta \quad \text{and} \quad d_2 = (-2\Delta\phi_{12} + f_Z\phi_{11Z})/\mathrm{i}\alpha\lambda. \tag{3.8b}$$

(*b*) *The largest forced fundamental*

Let us now consider the largest forced term of the fundamental (due to non-neutrality effects), denoted by $(\hat{u}_2^{(1)}, \hat{v}_2^{(1)}, \hat{w}_2^{(1)}, \hat{p}_2^{(1)})$. It is found that

$$\frac{\partial}{\partial Y}\left[\frac{\hat{p}_{2Y}^{(1)}}{(\bar{u}_0 - c)^2}\right] + \frac{\partial}{\partial Z}\left[\frac{\hat{p}_{2Z}^{(1)}}{(\bar{u}_0 - c)^2}\right] - \frac{\alpha^2\hat{p}_2^{(1)}}{(\bar{u}_0 - c)^2} = \frac{R_2}{(\bar{u}_0 - c)^2}, \tag{3.9a}$$

with boundary conditions

$$\hat{p}_{2Y}^{(1)}(Y = 0) = 0, \quad \hat{p}_2^{(1)}(Y \to \infty) = 0, \tag{3.9b}$$

and

$$R_2 = R_2^{(1)}\frac{\partial A}{\partial \tilde{X}} + R_2^{(0)}\tilde{x} + m^{-2}\tilde{X})A, \tag{3.9c}$$

where

$$R_2^{(1)} = \left[\frac{c\mathrm{i}}{\alpha(\bar{u}_0 - c)}(\hat{p}_{1YY}^{(1)} + \hat{p}_{1ZZ}^{(1)} - \alpha^2\hat{p}_1^{(1)}) - 2\mathrm{i}\alpha\hat{p}_1^{(1)}\right], \tag{3.9d}$$

$$R_2^{(0)} = -\left[\frac{\bar{u}_1}{(\bar{u}_0 - c)}(\hat{p}_{1YY}^{(1)} + \hat{p}_{1ZZ}^{(1)} - \alpha^2\hat{p}_1^{(1)}) - \frac{2}{(\bar{u}_0 - c)}(\bar{u}_{1Y}\hat{p}_{1Y}^{(1)} + \bar{u}_{1Z}\hat{p}_{1Z}^{(1)})\right]. \tag{3.9e}$$

Note that the equation for $\hat{p}_2^{(1)}$ is an inhomogeneous form of that for $\hat{p}_1^{(1)}$; as mentioned previously, we must determine a solvability condition for this equation to ensure that it has a solution. This solvability condition will (indirectly) lead to the desired evolution equation. To derive the solvability condition we will essentially follow the conventional method of multiplying the inhomogeneous equation by the adjoint of the homogeneous equation and then integrate over the range $0 \leqslant Y \leqslant \infty$, $0 \leqslant Z \leqslant 2\pi$ (the latter corresponding to a complete period). Note that we are assuming that the disturbance has the same period as the basic flow. The latter assumption is justified by the fact that Hall & Horseman (1991) were unable to find any subharmonic disturbances of the modified Rayleigh pressure equation. As in all critical-layer problems, special care must be taken to deal with the singular nature of the equations, i.e. the critical layers; however, extra care is necessary for the current problem due to the 'waviness' (Z-dependence of $Y_c \equiv f(Z)$) of the critical layer. The fact that additional care is required can be explained as follows: note that, in (Y, Z)-coordinates, (i) for any fixed Z-value there is a unique critical-Y value, but (ii), for a fixed Y-value there is either no critical Z-values or many.

Thus, we look for coordinates which describe the critical level more suitably; note that the Prandtl transformation ($s = Y - f(Z)$) does 'level' out the critical layer but, at the same time, leaves the solid boundary 'wavy', i.e. it is described by $s = -f(Z)$. Instead we introduce the normal variable

$$\xi = Y/f(Z). \tag{3.10}$$

This transformation 'flattens' out the critical level (it corresponds to $\xi = 1$) and also leaves the wall flat ($\xi = 0$). Note that this transformation involves $f(Z)$ and thus could not have been used from the outset, i.e. it involves information from the solution of the homogeneous problem (2.6).

Note that

$$\partial/\partial Y \to (1/f(Z))\partial/\partial\xi, \quad \partial/\partial Z \to \partial/\partial Z + g(Z)\xi\partial/\partial\xi, \tag{3.11}$$

where $g = -f'/f$. In these coordinates, the homogeneous equation takes the form

$$\frac{1}{f^2}\frac{\partial}{\partial\xi}\left[\frac{\hat{p}^{(1)}_{1\xi}}{(\bar{u}-c)^2}\right] + \left(\frac{\partial}{\partial Z} + g\xi\frac{\partial}{\partial\xi}\right)\left[\frac{(\partial/\partial Z + g\xi\partial/\partial\xi)\hat{p}^{(1)}_1}{(\bar{u}-c)^2}\right] - \frac{\alpha^2\hat{p}^{(1)}_1}{(\bar{u}-c)^2} = 0, \tag{3.12a}$$

$$\hat{p}^{(1)}_{1\xi}(\xi = 0) = 0, \quad \hat{p}^{(1)}_1(\xi = \infty) = 0, \tag{3.12b}$$

while the adjoint, q say, satisfies

$$\frac{1}{f^2}\frac{\partial}{\partial\xi}\left[\frac{q_\xi}{(\bar{u}-c)^2}\right] + \left(\frac{\partial}{\partial Z} - g\left(\frac{\partial}{\partial\xi}\right)\xi\right)\left[\frac{(\partial/\partial Z - g(\partial/\partial\xi)\xi)q}{(\bar{u}-c)^2}\right] - \frac{\alpha^2 q}{(\bar{u}-c)^2} = 0, \tag{3.13a}$$

$$q_\xi(\xi = 0) = 0, \quad q(\xi = \infty) = 0. \tag{3.13b}$$

Therefore, in these transformed coordinates the pressure equation is not self-adjoint.

Thus, to derive the solvability condition for (3.9), we first transform to (ξ, Z) variables, multiply both sides by the adjoint q (defined above) and then integrate both sides over the range $0 \leqslant \xi \leqslant \infty$, $0 \leqslant Z \leqslant 2\pi$, *excluding the critical layer*. This gives

$$\int_0^{2\pi} \mathrm{d}Z \left[\frac{1}{(\bar{u}-c)^2}\left(\xi g(q\hat{p}^{(1)}_{2Z} - \hat{p}^{(1)}_2 q_Z) + \frac{1}{f^2}(q\hat{p}^{(1)}_{2\xi} - \hat{p}^{(1)}_2 q_\xi)\right.\right.$$
$$\left.\left. + g^2(\xi^2 q\hat{p}^{(1)}_{2\xi} - \xi\hat{p}^{(1)}_2(\xi q)_\xi)\right)\right]_{\xi=1^-}^{\xi=1^+} = -\int_0^{2\pi}\!\!\!\!\!\!\!-\!\!\int_0^{\infty} \frac{qR_2}{(\bar{u}-c)^2}\,\mathrm{d}\xi\,\mathrm{d}Z, \tag{3.14}$$

where the bar through the integral represents the finite part, i.e. excluding critical-layer effects. Note that the modified Rayleigh pressure equation (2.6a), (3.12a) is a *partial* differential equation, and, thus, so is the adjoint equation (3.13*a*); hence, the need to integrate over both ξ and Z above.

Near the critical level, q is of the form

$$q = q_{10} + q_{11}(\xi - 1) + q_{12}(\xi - 1)^2 + 0(\xi - 1)^3 \ln(\xi - 1) + q_{13}(\xi - 1)^3 + \cdots,$$

while $R_2^{(0)}, R_2^{(1)}$ have the forms

$$R_2^{(k)} = r_{-1}^{(k)}(\xi - 1)^{-1} + r_0^{(k)} + r_1^{(k)}(\xi - 1) + r_2^{(k)}(\xi - 1)^2 + \cdots, \quad k = 0, 1,$$

(the expressions for the coefficients are simply obtained by expanding (3.9*d*) and (3.9*e*) near the critical layer, however they are somewhat long and for the sake of brevity are not given here) while $\hat{p}_2^{(1)}$ has the form

$$\hat{p}_2^{(1)} = a_2\phi_1 + b_{2\pm}\phi_2 + \hat{p}_{2\mathrm{PI}}^{(1)}.$$

Here the first two terms correspond to solutions of the homogeneous equation, $a_2, b_{2\pm}$ are functions of Z and subscript PI denotes particular integral. The jump $b_{2+} - b_{2-}$ will be determined in the subsequent analysis, but it is not necessary to determine a_2. In fact, we define $a_2, b_{2\pm}$ so that $\hat{p}_{2\mathrm{PI}}^{(1)}$ has no terms in $(\xi - 1)^0, (\xi - 1)^3$, i.e. $\hat{p}_{2\mathrm{PI}}^{(1)}$ is properly defined. Then, near the critical layer, $\hat{p}_{2\mathrm{PI}}^{(1)}$ has the form

$$\hat{p}_{2\mathrm{PI}}^{(1)} = \beta_1(\xi - 1) + \beta_2(\xi - 1)^2 + \beta_{3\mathrm{L}}(\xi - 1)^3 \ln(\xi - 1) + 0(\xi - 1)^3 + \cdots,$$

where

$$\beta_{3\mathrm{L}} = \beta_{3\mathrm{L}}^{(1)} \frac{\partial A}{\partial \tilde{X}} + \beta_{3\mathrm{L}}^{(0)}(\tilde{x} + m^{-2}\tilde{X})A,$$

and

$$\beta_{3\mathrm{L}}^{(k)} = \frac{1}{\lambda^2}(q_{12}r_{-1}^{(k)} + q_{11}r_0^{(k)} + q_{10}r_1^{(k)}) - \frac{1}{\lambda^3}(q_{11}r_{-1}^{(k)}\lambda_2 + q_{10}r_0^{(k)}\lambda_2) + \frac{q_{10}r_{-1}^{(k)}}{2\lambda^2}\left(\frac{3\lambda_2^2}{2\lambda^2} - \frac{2\lambda_3}{3\lambda}\right), \quad k = 0, 1.$$

Thus, the solvability condition becomes

$$\int_0^{2\pi} \frac{3\Delta q_{10}}{f^2\lambda^2}(b_{2+} - b_{2-})\,\mathrm{d}Z = -\int_0^{2\pi} \fint_0^{\infty} \frac{qR_2}{(\bar{u} - c)^2}\,\mathrm{d}\xi\,\mathrm{d}Z. \tag{3.15}$$

Near the critical layer, the Prandtl-transformed normal velocity has the form

$$\begin{aligned}\hat{v}_{2\mathrm{P}}^{(1)} &\equiv \hat{v}_2^{(1)} - f_Z\hat{w}_2^{(1)}, \\ &= \pi_{-1}(\xi - 1)^{-1} + \pi_0 + \pi_{1\mathrm{L}}(\xi - 1)\ln(\xi - 1) \\ &\quad + \pi_{1\pm}(\xi - 1) + \cdots, \quad (\xi > 1),\end{aligned} \tag{3.16a}$$

where

$$\pi_{1\mathrm{L}} = -\frac{3\Delta}{\mathrm{i}\alpha\lambda}\beta_{3\mathrm{L}} \quad \text{and} \quad \pi_{1+} - \pi_{1-} = -\frac{3\Delta}{\mathrm{i}\alpha\lambda}(b_{2+} - b_{2-}). \tag{3.16b}$$

Note that for $\xi < 1$, the logarithm $\ln(\xi - 1)$ in (3.16*a*) must be replaced by $\ln|\xi - 1| - \mathrm{i}\pi$; the negative sign in front of $\mathrm{i}\pi$ follows from an inspection of the Stokes lines of the operator that occurs in the governing equations for the linear problem in the critical layer.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Thus, the quantity $b_{2+} - b_{2-}$ can be expressed in terms of the jump that $\hat{v}^{(1)}_{2\mathrm{P}\xi}$ suffers across the critical layer:

$$b_{2+} - b_{2-} = -\frac{\mathrm{i}\alpha\lambda}{3\Delta}\left[\hat{v}^{(1)}_{2\mathrm{P}\xi}\right]^{1+}_{1-} - \mathrm{i}\pi\beta_{3\mathrm{L}}. \tag{3.17}$$

The second term on the right-hand side corresponds to the linear part of the jump that $\hat{v}^{(1)}_{2\mathrm{P}\xi}$ suffers across the critical layer; the first term on the right-hand side corresponds to the nonlinear jump which is calculated in the following two sections. Once the nonlinear jump is determined, equation (3.17) will give the evolution equation for $A(\tilde{X})$.

4. The critical layer

Let us now consider the critical-layer flow (corresponding to region III in figure 3) with our aim to calculate expressions for the jumps in the solvability condition obtained in the previous subsection; we shall find that another region, the so-called diffusion layer, also needs to be considered and this is the subject of the next subsection. The analysis has similarities to that given in Appendix B of Hall & Smith (1991), Appendix A of Brown *et al.* (1993); and Wu (1993). Therefore, only the essential details of the analysis will be given here.

The critical layer is centred on $Y = Y_c \equiv f(x, Z)$ and has thickness $O(Re^{-1/6})$ relative to the boundary-layer thickness; it is a viscous critical layer. We introduce the critical-layer normal variable $\eta \sim O(1)$, where

$$Re^{1/2}\, y = Y = f(Z) + Re^{-1/6}\,\eta, \tag{4.1a}$$

so that now

$$\frac{\partial}{\partial Z} \to \frac{\partial}{\partial Z} - Re^{1/6}\, f_Z \frac{\partial}{\partial \eta}, \tag{4.1b}$$

and the transformed normal velocity v_P is defined by

$$v_P = v - f_Z w. \tag{4.1c}$$

The underlying three-dimensional flow has the form

$$\bar{u} = c + Re^{-1/6}\,\lambda(Z)\eta + Re^{-1/3}\,\lambda_2(Z)\eta^2/2 + \cdots + \mu\bar{u}_{10}(Z) + \mu\, Re^{-1/6}\,\bar{u}_{11}(Z)\eta + \mu\, Re^{-1/3}\,\bar{u}_{12}(Z)\eta^2/2 + \cdots. \tag{4.2}$$

Since we are looking at a viscous critical layer (the flow is marginally unstable), we must assume that

$$\mu \ll Re^{-1/6}.$$

However, depending on the size of μ, the above expansion (4.2) will be disordered.

In the critical layer, the perturbation to the three-dimensional underlying base flow has the following form:

$$(u, v_P, w, p) = \epsilon[(Re^{1/6}\,\hat{U}_1, \hat{V}_{\mathrm{P}1}, Re^{1/6}\,\hat{W}_1, \hat{P}_1)E + \mathrm{c.c.}] + \cdots + \epsilon^2(Re^{1/2}\,\hat{U}_0, Re^{1/3}\,\hat{V}_{\mathrm{P}0}, Re^{1/2}\,\hat{W}_0, Re^{1/3}\,\hat{P}_0)E^0 + \cdots. \tag{4.3}$$

Here only the terms of direct interest have been shown; the second harmonic does not contribute to the nonlinear jump and therefore has not been included above. Recall that the small parameter ϵ is a measure of the size of the linear inviscid secondary instability (see (2.5)); in § 5 we give a relationship between the small parameters ϵ, μ and the Reynolds number $Re \gg 1$, necessary for the nonlinear term in the amplitude evolution equation for $A(\tilde{X})$ to be 'competitive'.

(*a*) *The fundamental*

Writing

$$\hat{U}_1 = \hat{U}_1^{(1)} + Re^{-1/6}\,\hat{U}_1^{(2)} + \cdots, \tag{4.4}$$

with similar expansions for $\hat{V}_{\mathrm{P}1}$, $\hat{W}_1$ and $\hat{P}_1$, and substituting into the governing (Prandtl-transformed) equations leads to the following system for the leading-order fundamental:

$$\left.\begin{aligned} \mathrm{i}\alpha\hat{U}_1^{(1)} + \hat{V}_{P1\eta}^{(1)} + \hat{W}_{1Z}^{(1)} = 0, \quad \hat{P}_{1\eta}^{(1)} = 0,\\ \mathrm{i}\alpha\lambda\eta\hat{U}_1^{(1)} + \lambda\hat{V}_{\mathrm{P}1}^{(1)} + \lambda_Z\eta\hat{W}_1^{(1)} = -\mathrm{i}\alpha\hat{P}_1^{(1)} + \Delta\hat{U}_{1\eta\eta}^{(1)},\\ \mathrm{i}\alpha\lambda\eta\hat{W}_1^{(1)} = -\hat{P}_{1Z}^{(1)} + f_Z\hat{P}_{1\eta}^{(2)} + \Delta\hat{W}_{1\eta\eta}^{(1)}, \end{aligned}\right\} \tag{4.5a}$$

which must be solved subject to the condition that solutions match to their outer counterparts as $\eta \to \pm\infty$. The y-momentum equation of the next order problem (for $\hat{U}_1^{(2)}$, etc.) yields, in particular,

$$\hat{P}_{1\eta}^{(2)} = f_Z\hat{P}_{1Z}^{(1)}/\Delta, \tag{4.5b}$$

enabling a solution of (4.5*a*) to be determined. The solutions which match with the outer flow can be found in the paper by Brown *et al.* (1993); in our notation they are

$$\hat{P}_1^{(1)} = A(\tilde{X})\phi_{10}(Z),$$

$$\hat{W}_1^{(1)} = -\frac{1}{a^{2/3}\Delta^2}\frac{\partial\hat{P}_1^{(1)}}{\partial Z}\int_0^\infty \exp[-\mathrm{i}a^{1/3}\eta t - t^3/3]\,\mathrm{d}t,$$

$$\hat{U}_1^{(1)} = \frac{\mathrm{i}}{\alpha}\frac{\partial\hat{W}_1^{(1)}}{\partial Z} - \frac{\mathrm{i}}{4\alpha}\left(\frac{\lambda_Z}{\lambda} + \frac{a_Z}{a}\right)\frac{\partial}{\partial\eta}(\eta\hat{W}_1^{(1)}),$$

and

$$\hat{V}_{\mathrm{P}1}^{(1)} = -\left(\frac{\lambda_Z}{\lambda} + \frac{a_Z}{a}\right)\frac{\eta\hat{W}_1^{(1)}}{4} + d_2 - \frac{1}{2i\alpha\lambda\Delta}\left(\frac{\lambda_Z}{\lambda} - \frac{f_Z f_{ZZ}}{\Delta}\right), \tag{4.6}$$

where

$$a = \lambda\alpha/\Delta. \tag{4.7}$$

At this stage we also point out that $\lambda, \alpha, a, \Delta$ are all real.

It is also possible to consider further terms in the fundamental; in fact, by considering the terms due to the non-neutrality, it is possible to derive the logarithmic jump effect deduced earlier from the asymptotic form of the outer solution as the critical layer is approached. As a number of different orders have to be considered, and the answers are algebraically messy, we do not present that analysis here.

(*b*) *The zeroth harmonic*

We now consider the largest zeroth harmonic, or mean-flow correction, (proportional to E^0) due to the nonlinear interactions of the fundamental. Writing

$$\hat{U}_0 = \hat{U}_0^{(1)} + Re^{-1/6}\,\hat{U}_0^{(2)} + \cdots, \tag{4.8}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

with similar expansions for $\hat{V}_{\mathrm{P}0}, \hat{W}_0$ and $\hat{P}_0$, and substituting into the governing (Prandtl-transformed) equations leads to the following system for the leading-order zeroth harmonic

$$\left.\begin{array}{l} \hat{V}^{(1)}_{P0\eta} + \hat{W}^{(1)}_{0Z} = 0, \quad \Delta\hat{U}^{(1)}_{0\eta\eta} - \lambda\hat{V}^{(1)}_{\mathrm{P}0} - \lambda_Z\eta\hat{W}^{(1)}_{0} = S_1, \quad \hat{P}^{(1)}_{0\eta} = 0, \\ f_Z\hat{P}^{(1)}_{0Z} - \Delta\hat{P}^{(2)}_{0\eta} - 2f_{ZZ}\hat{W}^{(1)}_1\hat{W}^{(1)*}_1 = 0, \quad \Delta\hat{W}^{(1)}_{0\eta\eta} + f_Z\hat{P}^{(2)}_{0\eta} - \hat{P}^{(1)}_{0Z} = S_2, \end{array}\right\} \quad (4.9a)$$

where, in particular,

$$S_2 = \mathrm{i}\alpha\hat{U}^{(1)*}_1\hat{W}^{(1)}_1 + \hat{V}^{(1)*}_{\mathrm{P}1}\hat{W}^{(1)}_{1\eta} + \hat{W}^{(1)*}_1\hat{W}^{(1)}_{1Z} + \mathrm{c.c.} \quad (4.9b)$$

As usual in such studies, we choose $\hat{P}^{(1)}_0 = 0$ (any non-zero choice would just correspond to a different mean flow). Thus,

$$\Delta\hat{W}^{(1)}_{0\eta\eta} = S_2 + 2\frac{f_Z f_{ZZ}}{\Delta}\hat{W}^{(1)}_1\hat{W}^{(1)*}_1, \quad (4.10a)$$

from which it follows that

$$\hat{W}^{(1)}_0 \to [\pm F_w + C]\eta, \quad \hat{V}^{(1)}_{\mathrm{P}0} \to -[\pm F_w + C]_Z\eta^2/2, \quad (4.10b)$$

and

$$\hat{U}^{(1)}_0 \to \frac{1}{24\Delta\lambda}\left(\lambda^2[\pm F_w + C]\right)_Z\eta^4, \quad \text{as} \quad \eta\to\infty, \quad (4.10c)$$

where

$$F_w(\tilde{X}, Z) = \left((|\phi_{10}|^2)_Z - \left(\frac{5a_Z}{3a} + \frac{7\Delta_Z}{2\Delta}\right)|\phi_{10}|^2\right)\frac{\pi(2/3)^{2/3}\Gamma(1/3)}{a^{5/3}\Delta^5}|A(\tilde{X})|^2, \quad (4.10d)$$

and it is important at this stage to note that the $\tilde{X}$ and Z-dependences of F_w are separable. Here, C is a function of $\tilde{X}$, Z and does not need to be determined in this analysis.

These asymptotes imply that the zeroth harmonic *grows* on leaving the critical layer and that it is necessary to consider another flow region (the diffusion layer) where diffusion effects can counteract this growth. In general, such diffusion layers need to be considered in all nonlinear analyses dealing with viscous critical layers (see, for example, Brown & Stewartson 1978; Wu 1993; cf. the buffer layer of weakly nonlinear vortex-wave theories).

5. The diffusion layer and the evolution equation

This layer (regions IIa,b in figure 3) is introduced to take care of the growing mean-flow corrections as the critical layer is left; it turns out that the nonlinear term in the desired evolution equation for A stems entirely from this region. It is necessary to introduce the new scaled normal coordinate

$$\tilde{\eta} = (Y - f(Z))/(\delta_2\Delta^{1/2}c^{-1/2}), \quad \delta_2 = Re^{-1/4}\mu^{-1/2} \ll 1, \quad (5.1a)$$

and so here the mean flow has the form

$$\bar{u} = c + (\Delta/c)^{1/2}\delta_2\lambda\tilde{\eta} + \cdots. \quad (5.1b)$$

The perturbation of this mean flow has the form

$$(u, v_P, w, p) = \epsilon[(\delta_2^{-1}\tilde{U}_1, \tilde{V}_{\mathrm{P}1}, \delta_2^{-1}\tilde{W}_1, \tilde{P}_1)E + \mathrm{c.c.}] + \cdots$$
$$+\epsilon^2(\delta_2^4 Re^{7/6}\tilde{U}_0, \delta_2^2 Re^{2/3}\tilde{V}_{\mathrm{P}0}, \delta_2 Re^{2/3}\tilde{W}_0, \delta_2^{-1}\tilde{P}_0)E^0 + \cdots. \quad (5.2)$$

The fundamental has the expansion

$$\tilde{U}_1 = \tilde{U}_1^{(1)} + \cdots + \delta_3 \tilde{U}_1^{(2)} + \cdots, \qquad \delta_3 = \epsilon^2 \delta_2^3 \, Re^{7/6}, \tag{5.3}$$

with similar series for $\tilde{V}_{\mathrm{P}0}$, $\tilde{W}_0$ and $\tilde{P}_0$. Note that the largest fundamental in this region is merely a continuation of the outer solution in the locality of the critical layer:

$$(\tilde{U}_1^{(1)}, \tilde{V}_{\mathrm{P}1}^{(1)}, \tilde{W}_1^{(1)}, \tilde{P}_1^{(1)}) = (c^{1/2} \Delta^{-1/2} d_{1Z} \tilde{\eta}^{-1}, d_2, -\mathrm{i}\alpha c^{1/2} \Delta^{-1/2} d_1 \tilde{\eta}^{-1}, \phi_{10}) A(\tilde{X}). \tag{5.4}$$

(*a*) *The leading zeroth harmonic*

Writing

$$(\tilde{U}_0, \tilde{V}_{\mathrm{P}0}, \tilde{W}_0) = (\tilde{U}_0^{(1)}, \tilde{V}_{\mathrm{P}0}^{(1)}, \tilde{W}_0^{(1)}) + \cdots,$$

we find that the governing equations for the leading zeroth harmonic are

$$\Delta^{-1/2} c^{1/2} \tilde{V}_{P0\tilde{\eta}}^{(1)} + \tilde{W}_{0Z}^{(1)} = 0, \tag{5.5a}$$

$$\left(\frac{\partial}{\partial \tilde{X}} - \frac{\partial^2}{\partial \tilde{\eta}^2} \right) \tilde{U}_0^{(1)} + \frac{\lambda}{c} \tilde{V}_{\mathrm{P}0}^{(1)} + \frac{\Delta^{1/2} \lambda_Z}{c^{3/2}} \tilde{\eta} \tilde{W}_0^{(1)} = 0, \tag{5.5b}$$

and

$$\left(\frac{\partial}{\partial \tilde{X}} - \frac{\partial^2}{\partial \tilde{\eta}^2} \right) \tilde{W}_0^{(1)} = 0. \tag{5.5c}$$

It follows, from matching with the critical-layer solutions, that the boundary conditions at $\tilde{\eta} = \pm 0$ are

$$\frac{\partial^4 \tilde{U}_0^{(1)}}{\partial \tilde{\eta}^4} = \frac{1}{\lambda c} (\lambda^2 F_\pm)_Z, \qquad \frac{\partial^2 \tilde{V}_{\mathrm{P}0}^{(1)}}{\partial \tilde{\eta}^2} = -F_{\pm Z}, \qquad \frac{\partial \tilde{W}_0^{(1)}}{\partial \tilde{\eta}} = \Delta^{-1/2} c^{1/2} F_\pm,$$

where $F_\pm = \Delta(\pm F_w + C)/c$. These equations can be solved using a Fourier-transform method to give

$$\tilde{U}_{0\tilde{\eta}\tilde{\eta}}^{(1)} = \pm (1/2c\pi^{1/2}) \tilde{\eta} \int_0^\infty t^{-1/2} \exp(-\tilde{\eta}^2/4t) \times (\lambda [F_\pm(\tilde{X} - t)]_Z + \tfrac{1}{4} \tilde{\eta}^2 t^{-1} \lambda_Z F_\pm(\tilde{X} - t) - \tfrac{3}{2} \lambda_Z F_\pm(\tilde{X} - t)) \, \mathrm{d}t, \tag{5.6a}$$

$$\tilde{V}_{P0\tilde{\eta}}^{(1)} = \pm \pi^{-1/2} \int_0^\infty t^{-1/2} [F_\pm(\tilde{X} - t)]_Z \exp(-\tilde{\eta}^2/4t) \, \mathrm{d}t, \tag{5.6b}$$

$$\tilde{W}_0^{(1)} = \mp (c/\pi\Delta)^{-1/2} \int_0^\infty t^{-1/2} F_\pm(\tilde{X} - t) \exp(-\tilde{\eta}^2/4t) \, \mathrm{d}t. \tag{5.6c}$$

As the mean-flow correction is larger in the diffusion layer than the critical layer, the leading-order nonlinear jump will result from the interaction of the mean flow and the largest fundamental in this (the diffusion) layer.

(*b*) *The largest forced fundamental*

The governing equations for the largest fundamental due to non-neutrality effects are

$$\mathrm{i}\alpha \tilde{U}_1^{(2)} + (c/\Delta)^{1/2} \tilde{V}_{P1\tilde{\eta}}^{(2)} + \tilde{W}_{1Z}^{(2)} = 0, \tag{5.7a}$$

$$\mathrm{i}\alpha(\Delta/c)^{1/2}\lambda\tilde{\eta}\tilde{U}_1^{(2)} + \lambda\tilde{V}_{\mathrm{P}1}^{(2)} + (\Delta/c)^{1/2}\lambda_Z\tilde{\eta}\tilde{W}_1^{(2)} + \mathrm{i}\alpha\tilde{P}_1^{(2)} = -\tilde{U}_0^{(1)}\tilde{U}_{1X}^{(1)} - (c/\Delta)^{1/2}\tilde{V}_{\mathrm{P}1}^{(1)}\tilde{U}_{0\tilde{\eta}}^{(1)} - \tilde{W}_1^{(1)}\tilde{U}_{0Z}^{(1)}, \tag{5.7b}$$

$$\tilde{P}_{1\tilde{\eta}}^{(2)} = 0, \tag{5.7c}$$

$$\mathrm{i}\alpha(\Delta/c)^{1/2}\lambda\tilde{\eta}\tilde{W}_1^{(2)} + \frac{1}{\Delta}\tilde{P}_{1Z}^{(2)} = -\tilde{U}_0^{(1)}\tilde{W}_{1X}^{(1)}. \tag{5.7d}$$

These can be combined to give an equation for $\tilde{V}_{\mathrm{P}1}^{(2)}$:

$$\tilde{\eta}\tilde{V}_{P1\tilde{\eta}\tilde{\eta}}^{(2)} = (c/\Delta)^{1/2}\lambda^{-1}A(\tilde{X})\frac{\partial}{\partial\tilde{\eta}}\left[-2\mathrm{i}\alpha d_1\tilde{\eta}^{-1}\left(\tilde{U}_{0Z}^{(1)} - \frac{\lambda_Z}{\lambda}\tilde{U}_0^{(1)}\right) + d_2\tilde{U}_{0\tilde{\eta}}^{(1)}\right]. \tag{5.7e}$$

Substituting for $\tilde{U}_0^{(1)}$ from equation (5.6) gives the required nonlinear jump

$$[\tilde{V}_{P1\tilde{\eta}}^{(2)}]_{-\infty}^{\infty} = \frac{A(\tilde{X})}{\lambda\Delta^{1/2}c^{1/2}}\left(d_2 - \frac{\mathrm{i}\alpha d_1}{\Delta}\left(\partial_Z - \frac{\lambda_Z}{\lambda}\right)\Delta\right)\int_0^\infty \lambda^2\left(\frac{F_w(\tilde{X}-\zeta,\,Z)}{\lambda}\right)_Z \mathrm{d}\zeta. \tag{5.8}$$

(c) The evolution equation

Noting that F_w is separable in $\tilde{X}$ and Z, we introduce $F_{ww}(Z)$ such that

$$F_w(\tilde{X}, Z) = F_{ww}(Z)|A(\tilde{X})|^2. \tag{5.9a}$$

Equation (5.8) may then be written in the form

$$[\tilde{V}_{P1\tilde{\eta}}^{(2)}]_{-\infty}^{\infty} = \frac{1}{\lambda\Delta^{1/2}c^{1/2}}\left(d_2 - \frac{\mathrm{i}\alpha d_1}{\Delta}\left(\partial_Z - \frac{\lambda_Z}{\lambda}\right)\Delta\right)\lambda^2 \times\left(\frac{F_{ww}(Z)}{\lambda}\right)_Z A(\tilde{X})\int_0^\infty |A(\tilde{X}-\zeta)|^2\,\mathrm{d}\zeta. \tag{5.9b}$$

The final evolution equation is obtained from (3.15), (3.17) and (5.9*b*). If the constants γ_1, γ_2 are defined by

$$\gamma_1 = \int_0^{2\pi}\left\{ ⨍_0^\infty \frac{qR_2^{(0)}}{(\bar{u}-c)^2}\mathrm{d}\xi - \frac{3\mathrm{i}\pi q_{10}\Delta\beta_{3\mathrm{L}}^{(0)}}{\lambda^2 f^2}\right\}\mathrm{d}Z \times\left[\int_0^{2\pi}\left\{ ⨍_0^\infty \frac{qR_2^{(1)}}{(\bar{u}-c)^2}\mathrm{d}\xi - \frac{3\mathrm{i}\pi q_{10}\Delta\beta_{3\mathrm{L}}^{(1)}}{\lambda^2 f^2}\right\}\mathrm{d}Z\right]^{-1}, \tag{5.10a}$$

$$\gamma_2 = \int_0^{2\pi}\left\{\frac{\mathrm{i}\alpha q_{10}}{c\lambda^2 f^3}\left(d_2 - \frac{\mathrm{i}\alpha d_1}{\Delta}\left(\partial_Z - \frac{\lambda_Z}{\lambda}\right)\Delta\right)\lambda^2\left(\frac{F_{ww}(Z)}{\lambda}\right)_Z\right\}\mathrm{d}Z \times\left[\int_0^{2\pi}\left\{ ⨍_0^\infty \frac{qR_2^{(1)}}{(\bar{u}-c)^2}\mathrm{d}\xi - \frac{3\mathrm{i}\pi q_{10}\Delta\beta_{3\mathrm{L}}^{(1)}}{\lambda^2 f^2}\right\}\mathrm{d}Z\right]^{-1}, \tag{5.10b}$$

then the required evolution equation is

$$\frac{\mathrm{d}A}{\mathrm{d}\tilde{X}} + \gamma_1(x_0)(\tilde{x} + m^{-2}\tilde{X})A = \sigma\gamma_2(x_0)A(\tilde{X})\int_0^\infty |A(\tilde{X}-\zeta)|^2\,\mathrm{d}\zeta, \quad A \to 0, \quad \text{as } \tilde{X} \to -\infty. \tag{5.11}$$

Here, the parameter

$$\sigma = \epsilon^2 \, Re^{2/3} \, / \mu^2 \tag{5.12}$$

measures the 'competitiveness' of the nonlinear term in the evolution equation, relative to the term corresponding to linear effects. For $\sigma \ll 1$, linear effects will dominate the evolution and the disturbance will continue to grow after $x = x_0$. Here we concern ourselves with the regime $\sigma \sim O(1)$, where linear and nonlinear effects have equal influence on the evolution of the disturbance. Recall that the parameter $m = \mu \, Re^{1/4}$ is a measure of the relative effect of the non-parallelism of the underlying three-dimensional flow on the evolution of the disturbance. Note that when $\mu \sim O(Re^{-1/4})$ the linear term in (5.11) is proportional to $\tilde{X} A$; however, for larger values of μ the linear term is, instead, proportional to A.

Thus, we see that the Z-dependence of the problem has been removed by the application of the solvability condition so that the Z-dependence of the nonlinear problem is non-local, as was found to be the case in the linear regime. Further, we see that our weakly nonlinear analysis has lead to a cubic nonlinearity; however, rather than appearing as a polynomial (e.g., as $A|A|^2$ if the evolution was described by the Stuart–Watson method), the nonlinear term is a convolution. The evolution equation is an integro-differential equation which depends on the entire history of the disturbance. Such evolution equations were first derived/proposed by Hickernell (1984). In fact Wu (1993) and Smith *et al.* (1993) have derived essentially the same equation in their studies of boundary-layer transition; however, as we have considered a fully three-dimensional boundary layer, our coefficients γ_1, γ_2 are far more complicated. Similar equations have also appeared in other recent papers (se, for example, Smith & Walton 1989; Smith & Blennerhassett 1992; Wu *et al.* 1993; Blackaby 1994).

6. Solution of the evolution equation and conclusions

The nature of the solution of (5.11) depends crucially on the sign of the real part of γ_2 (defined by equation (5.10*b*)) and on the size of μ. Since the disturbance under investigation becomes unstable as it moves downstream, we know that the real part of γ_1 is negative. Without calculating the solution of the neutral leading-order eigenfunction and adjoint problems, we cannot say what is the sign of the real part of γ_2. We shall therefore discuss both possibilities and use results from experiments to suggest the most likely scenario; we consider the two cases $m \sim O(1)$ and $m \gg 1$ separately.

(*a*) *The evolution equation for* $m \gg 1$

When $m \gg 1$, the (still small) parameter μ is sufficiently large that the non-parallelism of the underlying flow does not enter the evolution equation. In this case a suitably rescaled version of (5.11) takes the form

$$\frac{\mathrm{d}B}{\mathrm{d}\bar{X}} = B \pm B(\bar{X}) \int_0^{\infty} B(\bar{X} - \zeta) \, \mathrm{d}\zeta, \quad B \to 0, \quad \text{as } \bar{X} \to -\infty. \tag{6.1}$$

Here B is real, positive and proportional to $|A|^2$ and the $\pm$ signs correspond, respectively, to the cases when the real part of γ_2 is positive and negative, respectively.

The solution of (6.1), which has the required upstream behaviour, is given by

$$B = 2\mathrm{e}^{\bar{X}} / (1 \mp \mathrm{e}^{\bar{X}})^2. \tag{6.2}$$

If we take the negative sign in (6.2), corresponding to the positive sign in equation (6.1), we see that a singularity develops after a finite distance. If the positive sign is taken in (6.2), B grows as $\bar{X}$ increases from $-\infty$ until it reaches a maximum and then decays exponentially to zero. At first sight this seems a rather curious fate for a disturbance which was initially unstable on the basis of linear theory. However, the integral term in (6.1) can be interpreted as the effect on the growth rate by the mean flow corrected by the upstream development of the instability. Thus, the mean-flow modification, which occurs in the early stages of the growth of the disturbance, adjusts the mean flow so that it is linearly stable further downstream.

(b) *The evolution equation for* $m \sim O(1)$

When $m \sim O(1)$, the small size of μ means that the non-parallelism of the underlying flow does enter the evolution equation. In this case a suitably rescaled version of (5.11) takes a form different from (6.1):

$$\frac{\mathrm{d}B}{\mathrm{d}\bar{X}} = \bar{X}B \pm B(\bar{X}) \int_0^{\infty} B(\bar{X} - \zeta)\,\mathrm{d}\zeta. \tag{6.3}$$

This must be solved subject to an upstream initial condition. Again, B is real, positive and proportional to $|A|^2$ and the $\pm$ signs correspond, respectively, to the cases when the real part of γ_2 is positive and negative, respectively.

It is not possible to solve equation (6.3) analytically; however, it is possible to consider the possible large-$\bar{X}$ forms of the solutions analytically. We find that two possible large-$\bar{X}$ behaviours are possible: one in which B decays to zero, and another in which B grows as $\bar{X}$ increases until a singularity develops after a finite distance. The latter large-$\bar{X}$ possibility can only occur for the positive sign in equation (6.3). A numerical solution of equation (6.3) confirms these findings and also suggests that solutions for the case corresponding to the positive sign in equation (6.3) will always develop a singularity after a finite distance.

7. Conclusions

Experimental observations certainly suggest that the mode identified by Hall & Horseman (1991) continues to grow after it first becomes unstable. Here, we are assuming that the mode of instability discussed by Hall & Horseman (1991) is responsible for the experimentally observed onset of three-dimensionality in the Görtler problem. The closeness of the theoretically predicted most-unstable wavenumber and frequency with those measured by Swearingen & Blackwelder (1987) gives some backing for that assumption. The experiments therefore suggest that the negative sign is appropriate in (6.1), (6.3). However, without numerical solutions of the neutral eigenfunction and its adjoint, we cannot confirm that assertion.

We have carried out a viscous critical-layer analysis for a marginally unstable inviscid disturbance to a flow containing a streamwise vortex structure. The vortex structure could be the result of a centrifugal instability, wave interactions or other mechanisms. In fact, our analysis is valid for any flow where one of the velocity components depends on two spatial variables and is larger than the the other two components. For such more general flows, the periodicity in the spanwise direction, which we assume in this paper, must be replaced by an appropriate condition in order to derive the required solvability condition.

Our analysis is similar to that of Wu (1993) and Smith *et al.* Brown (1993); however, our analysis is complicated due to the fact that the nonlinear vortex state has rendered the boundary-layer flow three dimensional. As a consequence, our disturbances have a general Z-dependence, whereas Wu (1993) was able to consider separate harmonics in Z and derive coupled amplitude equations. In the problem considered by Smith *et al.* (1993), the initial boundary-layer flow is two dimensional; all subsequent vortex activity (three dimensionality of the boundary layer) is due to the relatively large mean-flow corrections induced in the diffusion (buffer) layers via nonlinear-interaction effects.

In this paper we have described the evolution of the inviscid modes found by Hall & Horseman (1991); this has achieved using viscous-critical-layer and diffusion-layer theories in the context of a weakly nonlinear instability theory. In particular, we have considered the evolution of a mode near the critical streamwise location, where the vortex structure has developed sufficiently to (first) render the now three-dimensional boundary-layer flow unstable to inviscid modes. At such a location, the flow is marginally unstable and we can consider the evolution of the most dangerous (important) mode. We note that our theory is not directly applicable to modes excited at streamwise locations where the flow supports a band of unstable modes (i.e. at an $O(1)$ distance downstream from the critical x-location); in such cases the most dangerous mode has too large a growth rate and the wavenumber will not be close enough to a 'neutral' value for weakly-nonlinear theory to be immediately applicable. However, it can be argued that viscous spreading effects (or some other external effect) will reduce the growth rates to a size where a weakly nonlinear theory (based on unsteady critical-layer theory rather than viscous critical-layer theory) is appropriate. The papers by, for example, Michalke (1964), Crighton & Gaster (1976) and Hultgren (1992) support such an argument, which has been used in many recent papers concerned with flow stability (see, for example, Goldstein & Leib 1988, 1989; Goldstein & Hultgren 1988; Hultgren 1992; Wu *et al.* 1993). The evolution of the Hall–Horseman modes for the non-marginal stability case is the subject of current study by the authors and will be reported in due course.

Our analysis shows that the disturbance amplitude satisfies the integro-differential equation (5.11/6.1). Experimental observations show that the linear growth of three-dimensional disturbances to Görtler vortices is rapidly followed by the onset of turbulence. Such a scenario would be consistent with (6.1) if the positive sign were taken in that equation. The excellent agreement between the experimental measurements of Swearingen & Blackwelder (1987) and Hall & Horseman (1991) for the linear regime lead us to believe that this is indeed the case. However, it is possible that the sign to be taken in (6.1) and (6.2) depends on the wavenumber and frequency of the marginally unstable mode and that some disturbances are destroyed by viscous effects. Therefore, it is conceivable that the linearly growing disturbances are inhibited by viscosity for weak vortex states and grow explosively further downstream when the vortex state has been reinforced.

Following the explosive growth of the disturbance, new effects must come into play and viscosity will play a secondary role; see Wu *et al.* (1993), who were concerned with the growth of inviscid disturbances to Stokes layers. The extension of our work along the line followed by the latter, and indeed other authors, is made non-trivial by the fact that in our calculation the critical layer is not flat.

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